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Shooting and Parallel Shooting Methods for Solving the Falkner-Skan Boundary-Layer Equation*

TUNCER CEBECI[†] AND HERBERT B. KELLER[‡]

Douglas Aircraft Company and California Institute of Technology

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We present three accurate and efficient numerical schemes for solving the Falkner-Skan equation with positive or negative wall shear. Newton's method is employed, with the aid of the variational equations, in all the schemes and yields quadratic convergence. First, ordinary shooting is used to solve for the case of positive wall shear. Then a nonlinear eigenvalue technique is introduced to solve the inverse problem in which the wall shear is prescribed and the pressure distribution is to be determined. With this approach the reverse flow solutions (i.e., negative wall shear) are obtained. Finally, a parallel shooting method is employed to reduce the sensitivity of the convergence of the iterations to the initial estimates.

1. INTRODUCTION

Laminar boundary layers exhibiting similarity have long been the subject of numerous studies since they play an important role in illustrating the main physical features of boundary-layer phenomena. They also provide a basis for approximate methods of calculating more complex, nonsimilar flows. In the case of two-dimensional flows, when the external velocity at the edge of the boundary layer, u_e, is of the form $u_e \sim x^{(\beta/2-\beta)}$, the equations of the incompressible laminar boundary layer become similar and can be reduced to

$$f''' + ff'' + \beta[1 - (f')^2] = 0. \tag{1}$$

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[†] California State College, Long Beach, Calif., and Douglas Aircraft Company, Long Beach, Calif.

[‡] Applied Mathematics, Firestone Laboratories, California Institute of Technology, Pasadena, Calif. 91109.

The usual boundary conditions are

$$f(0) = f'(0) = 0, \tag{2a}$$

$$\lim_{\eta_\infty \to \infty} f'(\eta_\infty) = 1. \tag{2b}$$

Equation (1) is the well-known Falkner–Skan equation, which has provided many fruitful sources of information about the behavior of incompressible boundary layers. Its solutions have been extensively studied and reported in the literature for various values of β. Most of these studies have concentrated on accelerating ($\beta > 0$), constant ($\beta = 0$), and decelerating ($\beta < 0$) flows ahead of the separation point (i.e., the point of zero wall shear). For all flows ahead of the separation point the wall shear, which is proportional to $f''(0)$, is greater than zero. However, physically relevant solutions exist only for values of β in the range of $-0.19884 \leqslant \beta \leqslant 2$. Zero wall shear corresponds to $\beta = -0.19884$. Flows for which the wall shear is less than zero are called reverse flows and correspond to flows beyond the separation point. They were first obtained by Stewartson [1]. These solutions exist only for β in the range : $-0.19884 < \beta < 0$. Thus there are two physically relevant solutions of (1), (2), in this latter β-range.

Equations (1) and (2) form a third-order nonlinear two-point boundary value problem for which no closed-form solutions are known. Thus numerical methods are usually employed and of these the most popular is the shooting method. This consists in solving an initial-value problem for (1) in which one keeps $f(0)$ and $f'(0)$ fixed at their proper values (zero in this case) and tries various values of $f''(0)$ in order to satisfy (2b). The systematic method by which new values of $f''(0)$ are determined is one of the main features of this paper and it is found to be far better than the usual "cut-and-try" methods that have been applied [2]. In Fig. 1 solutions of the initial value problems are illustrated for values of $f''(0)$ in the neighborhood of the "exact" value required to satisfy $f'(\infty) = 1$. For β negative, it has been observed that solutions for all values of $f''(0)$, sufficiently near the correct one, meet the proper boundary conditions at infinity. However, the desired solution, as described in Ref. [2] and [3], is the one that approaches $f' = 1$ most rapidly from below, as indicated in Fig. 1a. The other solutions for negative values of β have no apparent physical meaning. For β positive, solutions for all values of $f''(0)$ but the correct value diverge as shown in Fig. 1b.

The present paper utilizes the shooting methods as developed and described in [4]. For the simple shooting method the "cut-and-try" searching technique is replaced by Newton's method. This generally provides quadratic convergence of the iterations and decreases the computation time. In Section 2 we describe this application to the Falkner–Skan equation for positive wall shear values. It is found that Newton's method as we employ it automatically determines the physically relevant solution for negative β values.

For reverse flows, with negative wall shear, we proceed in another way; solving what may be termed a nonlinear eigenvalue problem [4]. That is, we fix $f''(0)$ to be the desired negative wall shear value and determine the appropriate value for β by iterations. This procedure is described in Section 3 and again Newton's method is used in conjunction with the shooting method. Quadratic convergence was usually obtained. It should be noted that this type of approach also becomes important in problems (usually in nonsimilar flows) in which pressure distribution for a prescribed wall shear is to be found; i.e., so-called inverse problems.

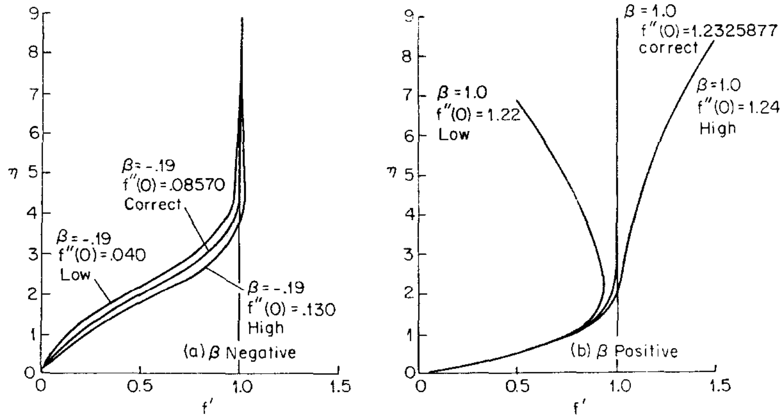

FIG. 1. Typical solutions of the Falkner–Skan boundary-layer equation.

One difficulty in both of the above applications is that the initial estimate of $f''(0)$ or of β must occasionally be very close to the exact value in order for the method to converge. These difficulties can be largely eliminated by employing the parallel shooting method [4]. We illustrate this in Section 4 only for flows with $f''(0) > 0$. Newton's method is now found to be extremely insensitive to the accuracy of the initial estimates and quadratic convergence is always observed.

2. Simple Shooting with Newton's Method. Positive Wall Shear

We first replace (1) by a system of three first-order ordinary differential equations. If the unknowns f, f', and f'' are denoted by f, u, and v, respectively, the system of three first-order equations can be written as

$$f' = u, \tag{3}$$

$$u' = v, \tag{4}$$

$$v' = -fv - \beta(1 - u^2). \tag{5}$$

In vector form this system can be written as

$$\mathbf{y}' = \mathbf{g}(\mathbf{y}), \tag{6}$$

where

$$\mathbf{y} = \begin{pmatrix} f \\ u \\ v \end{pmatrix} \qquad \mathbf{g}(\mathbf{y}) \equiv \begin{pmatrix} u \\ v \\ -fv - \beta(1 - u^2) \end{pmatrix}. \tag{7}$$

The boundary conditions given in (2) are replaced by

$$f(0) = 0, \qquad u(0) = 0 \tag{8a}$$

$$u(\eta_\infty) = 1. \tag{8b}$$

Here η_∞ is some "sufficiently large" value which is easily determined in the calculations. It varies with β but this aspect of the problem will not be discussed further. We denote the value of wall shear, $v(0)$, by

$$v(0) = s. \tag{8c}$$

The problem is to find s such that the solution of the initial value problem (6) and (8a, c) satisfies the outer boundary condition (8b). That is, if we indicate the solution of this initial value problem by $[f(\eta, s), u(\eta, s), v(\eta, s)]$ then we seek s such that

$$u(\eta_\infty, s) - 1 \equiv \varphi(s) = 0. \tag{9a}$$

To solve (9a) we employ Newton's method [5]. For some initial estimate s^0 of the root this yields the iterates s^ν defined by

$$s^{\nu+1} = s^\nu - \frac{\varphi(s^\nu)}{[d\varphi(s^\nu)/ds]} \equiv s^\nu - \frac{u(\eta_\infty, s^\nu) - 1}{[\partial u(\eta_\infty, s^\nu)/\partial s]} \; \nu = 0, 1, 2, \ldots. \tag{9b}$$

In order to obtain the derivative of u with respect to s, we take the derivatives of (6), (8a), and (8c) with respect to s. This leads to the following linear differential equations, known as the variational equations for (3), (4) and (5):

$$F' = U, \tag{10}$$

$$U' = V, \tag{11}$$

$$V' = -fV - vF + 2u\beta U; \tag{12}$$

and to the initial conditions

$$F(0) = 0, \qquad U(0) = 0, \qquad V(0) = 1. \tag{13}$$

Here

$$F \equiv \frac{\partial}{\partial s}, \qquad U \equiv \frac{\partial u}{\partial s}, \qquad V \equiv \frac{\partial v}{\partial s}. \tag{14}$$

Once the initial-value problem given by (6), (8a), and (10) through (13) are solved, $u(\eta_\infty, s^\nu)$ and $U(\eta_\infty, s^\nu) = \partial u(\eta_\infty, s^\nu)/\partial s$ are known and consequently the next approximation to $v(0)$, namely, $s^{\nu+1}$ can be computed from (9b). We use a fourth-order Runge–Kutta method to solve the initial value problems. (More efficient schemes should be employed if possible but in practice we are at the mercy of our programmers).

The above procedure was used to study flows for which wall shear was greater than or equal to zero. Calculations[1] for various values of β with various initial estimates of s^0 showed that the method is quite effective.

The convergence properties of the iterations depend upon the value of $\beta > 0$, the initial estimates s^0 must be more accurate as β increases and conversely for the decelerating flows, $\beta < 0$. For example, at $\beta = 0.5$ the converged wall shear is $v(0) = 0.92768$. When $s^0 = 0.1$, 0.2, 0.3, and 0.4 the iterations diverge while for $s^0 = 0.5$ they converge. Similarly at $\beta = 1.0$ the correct wall shear is $v(0) = 1.23259$ and the values $s^0 < 0.8$ lead to divergence while $s^0 = 0.9$ yields convergence. Some details of the convergence rates are displayed in Table I. Obviously a good initial guess, s^0, for some value of β is obtained by employing the converged value, $v(0)$, for a close value of β. In this way we easily determined the "exact" solution given in [3] never using more than three iterations.

TABLE I

Some Iterations for Accelerating and Decelerating Flows

Iteration No. ν	$\beta = 0.5$ s^ν	$\beta = 1$ s^ν	$\beta = -0.05$ s^ν	$\beta = -0.10$ s^ν
0	0.50000	0.9	0.10000	0.10000
1	0.539332	1.697839	0.396211	0.355755
2	0.623504	1.351750	0.400320	0.319287
3	0.822382	1.244502	0.4003238	0.3192733
4	0.923408	1.232734	—	—
5	0.927675	1.232590	—	—
6	0.927680	—	—	—

[1] All calculations were single-precision on an IMB 360/65. The convergence test was: $|s^{\nu+1} - s^\nu| < 10^{-6}$.

3. Nonlinear Eigenvalue Problem. Negative Wall Shear

To obtain the reverse flow solutions, we solve the system (6), (8a), (8b), and (8c) as a "nonlinear eigenvalue" problem with β as the unknown parameter. That is, by (8c) the value $v(0)$ of the wall shear is specified and we seek the appropriate value of β, the pressure gradient parameter. To do this we again employ shooting techniques and Newton's method. Specifically, we solve the initial value problem (6), (8a), and (8c) with $v(0) = s$ fixed and seek to vary β so that (8b) is satisfied. That is, if we denote the solution of this initial value problem by $[f(\eta, \beta), u(\eta, \beta), v(\eta, \beta)]$, then we seek β such that

$$u(\eta_\infty, \beta) - 1 \equiv \Psi(\beta) = 0 \tag{15}$$

Newton's method applied to this equation yields the iterates β^ν defined by

$$\beta^{\nu+1} = \beta^\nu - \frac{\Psi(\beta^\nu)}{[d\Psi(\beta^\nu)/d\beta]} \equiv \beta^\nu - \frac{u(\eta_\infty, \beta^\nu) - 1}{[\partial u(\eta_\infty, \beta^\nu)/\partial\beta]}, \qquad \nu = 0, 1, \ldots \tag{16}$$

The derivative $\partial u/\partial\beta$ is now obtained from the solution of the variational equations

$$p' = r, \tag{17}$$

$$r' = q, \tag{18}$$

$$q' = -fq - vp - (1 - u^2) + 2\beta ur, \tag{19}$$

subject to the initial conditions

$$p(0) = 0, \qquad q(0) = 0, \qquad r(0) = 0, \tag{20}$$

$$p = \frac{\partial f}{\partial \beta}, \qquad r = \frac{\partial u}{\partial \beta}, \qquad q = \frac{\partial v}{\partial \beta}, \tag{21}$$

and the system (17)–(20) is obtained by differentiating (3)–(5), (8a) and (8c) with respect to β.

Again, the fourth-order Runge–Kutta method is used to obtain the solution of the systems given by (6), (8a), (8c), and (17)–(20). The Newton iterates are then evaluated as in (16). This procedure has been applied with equal success to both nonseparating flows, $v(0) > 0$, and reverse flows, $v(0) < 0$. However, we discuss here only the results for reverse flows. As in the previous approach, convergence was obtained when the initial guess, β^0, was "reasonably" close to the correct value, β^*. Table II shows a comparison of reverse-flow solutions obtained by the present method and those obtained by Stewartson [1].

TABLE II

Comparison of Reverse-Flow Solutions

Number of Iterations ν	Fixed Parameter $v(0)$	Initial Estimate β^0	Converged Value β^*	Stewartson [1]
5	0	−0.26	−0.198851	—
4	−0.001	−0.26	−0.198826	—
5	−0.04	−0.26	−0.196348	—
5	−0.097	−0.26	−0.180552	—
3	−0.097	−0.18	−0.180553	−0.18
3	−0.132	−0.16	−0.152118	−0.15
7	−0.132	−0.26	−0.079596	—
3	−0.141	−0.1	−0.101763	−0.1
2	−0.108	−0.05	−0.049745	−0.05
5	−0.097	−0.05	−0.040286	—
2	−0.074	−0.025	−0.024789	−0.025
5	−0.04	−0.01	−0.009162	—

Figure 2 presents a plot of the nondimensional velocity profiles for the reverse flows. The results show that as the singular point $\beta = 0$, is approached, the magnitude of the reverse flow velocity decreases and the boundary-layer thickness increases. In the region very close to $\beta = 0$, it is difficult to obtain solutions.

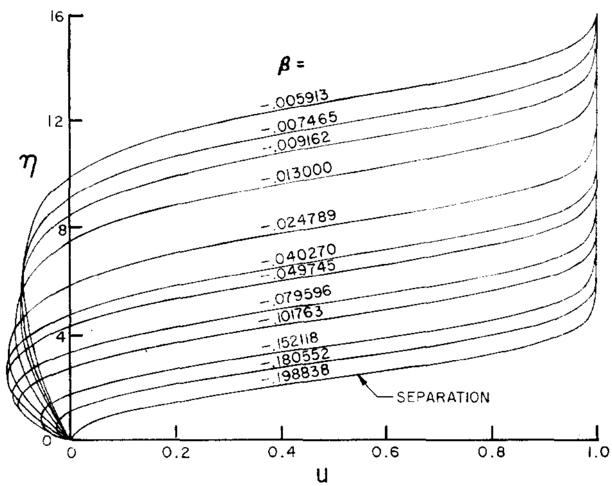

FIG. 2. Dimensionless velocity profiles for reverse flows.

4. Parallel Shooting, Positive Wall Shear

The sensitivity to the initial guess of the simple shooting methods, described in Sections 2 and 3 can be reduced by using the parallel shooting method [4]. According to this method, the total interval $[0, \eta_\infty]$ is divided into a number of subintervals, the appropriate initial-value problems are integrated over each subinterval, and then all of the "initial" data are adjusted *simultaneously* in order to satisfy the boundary conditions and appropriate continuity conditions at the subdivision points.

In the present study, we have arbitrarily divided the total transformed boundary-layer thickness η_∞ into three[2] subintervals: $[0, \eta^I]$, $[\eta^I, \eta^{II}]$, $[\eta^{II}, \eta_\infty]$. Over each subinterval the system (6) is solved subject to the initial conditions

$$\text{(I)}\ \mathbf{y}(0) = \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix}, \quad \text{(II)}\ \mathbf{y}(\eta^I) = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \quad \text{(III)}\ \mathbf{y}(\eta^{II}) = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}. \tag{22}$$

We denote the solutions of (6) over the subintervals I, II, III by $\mathbf{y}^I(\eta, s)$, $\mathbf{y}^{II}(\eta, a_1, b_1, c_1)$, and $\mathbf{y}^{II},(\eta, a_2, b_2, c_2)$, respectively. Then we impose the continuity conditions

$$\mathbf{y}^I(\eta^I, s) = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \mathbf{y}^{II}(\eta^I, a_1, b_1, c_1), \tag{23a}$$

$$\mathbf{y}^{II}(\eta^{II}, a_1, b_1, c_1) = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \mathbf{y}^{III}(\eta^{II}, a_2, b_2, c_2), \tag{23b}$$

and the boundary condition (8b) which becomes

$$(\mathbf{y}^{III})_2 \equiv u^{III}(\eta_\infty, a_2, b_2, c_2) = 1. \tag{23c}$$

This system of equations can also be written in vector form as

$$\boldsymbol{\varphi}(\mathbf{s}) \equiv \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{bmatrix} = \begin{bmatrix} f^I(\eta^I, s) - a_1 \\ u^I(\eta^I, s) - b_1 \\ v^I(\eta^I, s) - c_1 \\ f^{II}(\eta^{II}, a_1, b_1, c_1) - a_2 \\ u^{II}(\eta^{II}, a_1, b_1, c_1) - b_2 \\ v^{II}(\eta^{II}, a_1, b_1, c_1) - c_2 \\ u^{III}(\eta_\infty, a_2, b_2, c_2) - 1 \end{bmatrix} = 0 \tag{24a}$$

[2] The choice turned out to be quite satisfactory for the cases studied in this paper.

where, in transposed form,

$$\mathbf{s}^T = (s, a_1, b_1, c_1, a_2, b_2, c_2) \tag{24b}$$

The system (24a) has seven equations and seven unknowns. We solve this system by Newton's method which now yields the iterates $\mathbf{s}^\nu$ defined by:

$$\mathbf{s}^{\nu+1} = \mathbf{s}^\nu - \left[\frac{\partial \boldsymbol{\varphi}(\mathbf{s}^\nu)}{\partial \mathbf{s}}\right]^{-1} \boldsymbol{\varphi}(\mathbf{s}^\nu), \qquad \nu = 0, 1,.... \tag{25}$$

To find the Jacobian matrix $[(\partial\boldsymbol{\varphi}/\partial\mathbf{s})(\mathbf{s}^\nu)]$, we solve the following variational systems:

$$\begin{array}{lll}
\text{(I)} & 0 \leqslant \eta \leqslant \eta^I & \\
& F' = U & F^1(0) = 0, \\
& U' = V & U^1(0) = 0, \\
& V' = -f^I V - V^I F + 2\beta u^I U & V^1(0) = 1; \\
\text{(II)} & \eta^I \leqslant \eta \leqslant \eta^{II} & \\
& F' = U & F^2(\eta^I) = 1, \quad F^3(\eta^I) = 0, \quad F^4(\eta^I) = 0; \\
& U' = V & U^2(\eta^I) = 0, \quad U^3(\eta^I) = 1, \quad U^4(\eta^I) = 0; \\
& V' = -f^{II} V - V^{II} F + 2\beta u^{II} U & V^2(\eta^I) = 0, \quad V^3(\eta^I) = 0, \quad V^4(\eta^I) = 1; \\
\text{(III)} & \eta^{II} \leqslant \eta \leqslant \eta_\infty & \\
& F' = U & F^5(\eta^{II}) = 1, \quad F^6(\eta^{II}) = 0, \quad F^7(\eta^{II}) = 0; \\
& U' = V & U^5(\eta^{II}) = 0, \quad U^6(\eta^{II}) = 1, \quad U^7(\eta^{II}) = 0; \\
& V' = -f^{III} V - V^{III} F + 2\beta u^{III} U & V^5(\eta^{II}) = 0, \quad V^6(\eta^{II}) = 0, \quad V^7(\eta^{II}) = 1.
\end{array} \tag{26}$$

For example, in the first subinterval we solve the system of equations by using the given initial conditions in that subinterval and obtain the solution $F^1(\eta)$, $U^1(\eta)$, and $V^1(\eta)$. In the second subinterval, we solve the system of equations three times using the three sets of initial conditions with superscripts 2, 3, and 4. We denote these solutions by $F^j(\eta)$, $U^j(\eta)$, and $V^j(\eta)$ with $J = 2, 3$, and 4. Similarly, by solving the system of equations in the third subinterval using the three sets of initial conditions, 5, 6, and 7 we obtain the solutions $F^j(\eta)$, $U^j(\eta)$, and $V^j(\eta)$ with $j = 5, 6$, and 7. Using these solutions, the Jacobian matrix can be shown to be [4]

$$A^{(\nu)} \equiv \frac{\partial \boldsymbol{\varphi}(s^{(\nu)})}{\partial \mathbf{s}} \equiv \begin{bmatrix}
F^1(\eta^I) & -1 & 0 & 0 & 0 & 0 & 0 \\
U^1(\eta^I) & 0 & -1 & 0 & 0 & 0 & 0 \\
V^1(\eta^I) & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & F^2(\eta^{II}) & F^3(\eta^{II}) & F^4(\eta^{II}) & -1 & 0 & 0 \\
0 & U^2(\eta^{II}) & U^3(\eta^{II}) & U^4(\eta^{II}) & 0 & -1 & 0 \\
0 & V^2(\eta^{II}) & V^3(\eta^{II}) & V^4(\eta^{II}) & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & U^5(\eta_\infty) & U^6(\eta_\infty) & U^7(\eta_\infty)
\end{bmatrix} \tag{27}$$

In terms of this matrix the Newton iterates (25) are determined by solving the simple linear system:

$$A^{(\nu)}(\mathbf{s}^{\nu+1} - \mathbf{s}^{\nu}) = -\boldsymbol{\varphi}(\mathbf{s}^{\nu}). \tag{28}$$

In summary, our parallel shooting procedure proceeds by solving three initial value problems for (6) with initial data $\mathbf{s}^{\nu}$ as in (22). Along with these we solve the seven linear variational problems (26) to evaluate $A^{(\nu)}$ from (27). Finally, $\mathbf{s}^{\nu+1}$ is determined from (28) and one iteration cycle is completed.

Parallel shooting procedures were developed [4] to eliminate a shortcoming of simple shooting methods, frequently called "instability." This phenomenon is caused by the fact that rapidly growing solutions of the initial value problems magnify various errors (truncation as well as roundoff). Incorrect guesses at the appropriate unknown initial data are effectively truncation errors. Thus it may be expected that parallel shooting will reduce the sensitivity of the convergence of iteration procedures (like Newton's method) to the magnitude of the initial errors. This speculation was indeed borne out in the present calculations. Of course, the initial guess for parallel shooting is more complicated, in the present case requiring seven values rather than one. However, this additional complexity caused no difficulty and in the calculations presented below, we simply assumed a linear velocity profile from which all of the initial values (i.e., $\mathbf{s}^0$) were obtained by an integration and a differentiation. Tables III, IV, and V present the results obtained

TABLE III

Comparison of Calculated Results with Those of Reference [3]

	$v(0)$	
β	Parallel Shooting	Ref. [3]
−0.195	0.55177	0.055172
−0.19	0.085702	0.085700
−0.10	0.319278	0.319270
−0.05	0.400330	0.400323
0	0.469603	0.469600
0.10	0.587037	0.587035
0.20	0.686711	0.686708
0.40	0.854423	0.854421
0.60	1.120269	1.120268
1.00	1.232561	1.232588
1.20	1.335724	1.335772
1.60	1.521516	1.521514

TABLE IV

Convergence of $v(0)$; Parallel Shooting

Iteration No. ν	$\beta = 0.5$ $v(0)$	$\beta = 1.0$ $v(0)$	$\beta = -0.05$ $v(0)$
0	0.167	0.167	0.167
1	0.9445	2.23939	0.44785
2	0.927396	1.422649	0.399251
3	0.927683	1.243555	0.400330
4	0.927682	1.232634	—
5	—	1.232591	—

TABLE V

Convergence of **s** values for $\beta = 0.5$

Iteration No.	$v(0)$	a_1	b_1	c_1	a_2	b_2	c_2
0	0.167	0.25	0.5	0.167	0.5	1.0	0.167
1	0.9445	0.38951	0.698044	0.461947	2.302359	1.091020	0.083129
2	0.927396	0.380950	0.680833	0.444003	2.195862	0.994259	0.014278
3	0.927683	0.381092	0.681117	0.444287	2.197072	0.994964	0.014322
4	0.927682	0.381092	0.681116	0.444286	2.197067	0.994862	0.014324

in this manner. Table III gives a comparison of $v(0)$-values calculated by the parallel shooting method with those given in [3]. The agreement is very good and the disagreement which is in the sixth decimal place, is probably due to the round-off error since single precision arithmetic was used in the present calculation. In all these calculations the subintervals were arbitrarily taken as $0 \leqslant \eta \leqslant 1$, $1 \leqslant \eta \leqslant 3$ and $3 \leqslant \eta \leqslant 6$. The choice turned out to be satisfactory. When the subintervals were changed, almost identical results were obtained.

Table IV indicates the convergence of the iterations for three β values. These results, as for all other β values studied by parallel shooting, show excellent convergence properties. A comparison of Tables I and IV shows that parallel shooting has much less sensitivity to the initial guess.

A study was also made of the convergence of the components of $\mathbf{s}^\nu$ other than $s = v(0)$. Table V presents the results for $\beta = 0.5$. It is seen that the values of **s** other than $v(0)$ all converge much faster than $v(0)$ and this was typical. Note that for practical purposes convergence has occurred in the second iteration (i.e. three significant digits).

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